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STABILITY OF THE DISCONTINUITY FRONT IN MULTIPHASE MULTICOMPONENT DISPLACEMENT†

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Discontinuity from stability for multiphase multicomponent flows of compressible liquids and gases through a porous medium is analysed for multidimensional perturbations. A system of dispersion equations is derived for the stability of an arbitrary non-isothermal displacement front with any number of phases and components. Stability criteria are derived for perturbations which leave the discontinuity, and which have not been previously considered. The stability problem for a jump in concentration and water saturation when oil is being displaced by an active admixture solution is considered as an example.

Displacement stability for immiscible fluids of differing viscosities has been previously investigated for Hele–Shaw cells [1, 2] and porous media [3, 4], together with frontal stability in the Buckley–Leverett problem [5, 6]. More complicated problems of flow in porous media with interphase mass-exchange have also been considered [7]. However, only contact perturbations moving with the velocity of the discontinuity were taken into account there. A review of other approaches and investigations in frontal displacement stability is given in [8].

1. THE INITIAL SYSTEM OF EQUATIONS

The general system of equations describing multiphase multicomponent transfer in a porous medium in the large-scale approximations can be written in the form

$$\partial\rho_i/\partial t + \operatorname{div} \mathbf{j}_i = 0 \quad (i = 1, 2..., k) \tag{1.1}$$

$$\partial\rho_{k+1}/\partial t + \operatorname{div}\mathbf{j}_{k+1} = 0 \tag{1.2}$$

$$\mathbf{j}_i = -K_i(\rho_1,..., \rho_{k+1}, p)\, \nabla p \quad (i = 1, 2..., k) \tag{1.3}$$

Here ρ_i $(i=1, 2, \ldots, k)$ are the generalized number of phases (components), j_i are their fluxes, K_i is the generalized conductivity of the porous medium for the ith flux, and p is the pressure (ignoring the capillary pressure jump between phases). For example, in the Buckley–Leverett model [5] $\rho_1 = s$ is the saturation of the aqueous phase, $\rho_2 = 1-s$, $K_i = Kf_i(s)/\mu_i$ $(i=1, 2)$, where K is the permeability of the porous medium, and f_i, μ_i are the relative phase permeabilities and dynamic viscosities of the phases. In the general case the quantities K_i may depend on the pressure. One of the ρ_i may be taken to be the temperature. Here only

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convective heat transfer is included, and the effective thermal conductivity is ignored (the large-scale approximation [5]).

The given system is closed if there an additional relation exists between the functions ρ_i, ρ_{k+1} and p which can be interpreted as an equation of state

$$\rho_{k+1} = \varphi(\rho_1, ..., \rho_k, p) \tag{1.4}$$

System (1.1)–(1.4) can have discontinuous solutions. At a discontinuity the pressure continuity equation

$$[p] = 0 \tag{1.5}$$

and the Hugoniot condition (the law of conservation for ρ_i) [9]

$$[\rho_i]\, V_n = [j_{in}] \quad (i = 1, 2..., k + 1) \tag{1.6}$$

must be satisfied.

Here V_n is the normal component of the displacement front velocity and j_{in} $(i = 1, 2, \ldots, k+1)$ are the normal flux components.

We transform the original system so as to separate the "hyperbolic" part of the ρ_i transfer equations from the "elliptic" or "parabolic" pressure field. The "elliptic" ("parabolic") case corresponds to an incompressibility (compressibility) condition on the effective flux $\mathbf{j}$ (introduced below). We substitute Eq. (1.4) into (1.2). After reduction, we find

$$\varphi_i \frac{\partial \rho_i}{\partial t} + \varphi_p \frac{\partial p}{\partial t} + \operatorname{div} \mathbf{j}_{k+1} = 0 \ (i = 1, 2..., k); \ \ \varphi_i = \frac{\partial \varphi}{\partial \rho_i}, \ \varphi_p = \frac{\partial \varphi}{\partial p}$$

Here and below repeated indices are summed.

Substituting expressions for the derivatives $\partial\rho_i/\partial t$ obtained from (1.1) into the last equation, we reduce it to the form

$$\operatorname{div}\mathbf{j} + \mathbf{j}_n \nabla\varphi_n + \varphi_p \partial p/\partial t = 0, \quad \mathbf{j} = \mathbf{j}_{k+1} - \varphi_n \mathbf{j}_n \quad (n = 1, 2..., k) \tag{1.7}$$

The flux $\mathbf{j}$ is expressed from Eqs (1.3)

$$\mathbf{j} = -K\nabla p, \qquad K = K_{k+1} - \varphi_n K_n$$

Below we assume that by a suitable choice of fluxes $\mathbf{j}_i$ one can ensure that K never vanishes. The stronger assumption $K > 0$ corresponds to the physical idea that the porous medium resists the multicomponent flux. In this case the fluxes $\mathbf{j}_i$ can be expressed in terms of $\mathbf{j}$ by

$$\mathbf{j}_i = f_i \mathbf{j}\, j_i\, (\rho_1, ..., \rho_k, p) = K_i/K, \quad j_i = f_i j, \quad f_i(\rho_1, ..., \rho_k, p) = K_i/K \tag{1.8}$$

After substituting Eqs (1.7) and (1.8) into (1.1) we obtain a transformed system

$$\frac{\partial \rho_i}{\partial t} + A_{ij}(\mathbf{j}\nabla\rho_j) - f_i \varphi_p \frac{\partial p}{\partial t} - f_i B(\mathbf{j}\nabla p) = 0 \quad (i = 1, 2..., k)$$

$$\operatorname{div}\mathbf{j} + \mathbf{j}(b_j \nabla\rho_j) + \varphi_p \frac{\partial p}{\partial t} + B(\mathbf{j}\nabla p) = 0, \quad \mathbf{j} = -K\nabla p \tag{1.9}$$

$$b_j = f_n \frac{\partial \rho_n}{\partial \rho_j}, \quad B = f_n \frac{\partial \rho_n}{\partial p}, \quad A_{ij} = \frac{\partial f_i}{\partial \rho_j} - f_i b_j; \ (i, j = 1, 2..., k)$$

The nature of the transfer of the ρ_j is governed by the matrix $\mathbf{A}=[A_{ij}]$. "Hyperbolic" transfer means that the matrix $\mathbf{A}$ has k real and distinct eigenvalues $\lambda_1<\lambda_2<\ldots<\lambda_k$. It is assumed that this is satisfied, at least in some neighbourhood of the discontinuity under investigation.

We note that when passing through the discontinuity the ordering of the eigenvalues may change. Hence the same eigenvalue (as a function of hydrodynamic variables) could have different values before and after the discontinuity.

The more general case of non-strict hyperbolicity, when the eigenvalues may be identical can be similarly treated.

2. EQUATIONS IN THE NEIGHBOURHOOD OF THE DISPLACEMENT FRONT

Let O be any point on the front. We consider its motion at some instant of time (which without loss of generality can be zero). We direct the Ox axis along the normal to the front. We assume that its velocity at that instant is V and that the pressure at the point O is p_f.

We make the change of variables

$$x-Vt=\varepsilon x',\quad y=\varepsilon y',\quad z=\varepsilon z',\quad p-p_f=\varepsilon p' \tag{2.1}$$

We choose the parameter ε to be sufficiently small, which amounts to focusing on a neighbourhood of the point O. Ignoring terms of order ε and dropping the primes from the formulae, we reduce system (1.9) to the form

$$\frac{\partial\rho_i}{\partial t}+A_{ij}\mathbf{j}\nabla\rho_j-V\frac{\partial\rho_i}{\partial x}=0\quad(i=1,\ 2\ldots,\ k),\ \operatorname{div}\mathbf{j}+b_i(\mathbf{j}\nabla\rho_i)=0,\ \ \mathbf{j}=-K\nabla p \tag{2.2}$$

In the lowest approximation in ε the coefficients A_{ij}, b_j, and K do not depend on p and are determined by the pressure p_f.

We will describe the zeroth approximation in the neighbourhood of the front. We shall consider a frontal discontinuity: $\mathbf{j}_x=(j,0,0)$. Here, in the neighbourhood of the point O (with sufficiently small ε), the densities and fluxes in Eqs (2.2) can be assumed to be constant: $\rho_i=\rho_i^0=\rho_i^\pm$, $j_x=j_x^0=j_x^\pm$ (the minus and plus indices, respectively, denoting quantities before and after the discontinuity). The pressure field is then described by linear relations

$$\partial p^0/\partial x=j_x^0/K^0=\kappa^0=\kappa^\pm;\quad \partial p^0/\partial y=\partial p^0/\partial z=0 \tag{2.3}$$

The Hugoniot relations at the discontinuity are

$$[\rho_i^0]V=[f_i^0j_x^0]\ (i=1,\ 2\ldots,\ k),\quad [\rho_{k+1}^0]V=[j_x^0]+[\varphi_i^0f_i^0j_x^0],\quad [p^0]=0 \tag{2.4}$$

As well as relations (2.4) the discontinuity should satisfy additional stability conditions with respect to one-dimensional perturbations—the Lax conditions [9]. For one-dimensional stability it is necessary that $k+1$ hyperbolic characteristics of the system should arrive at the discontinuity. The corresponding system of inequalities can be written in the form

$$j_x^+\lambda_1^+<\ldots<j_x^+\lambda_l^+<V<j_x^-\lambda_l^-<\ldots<j_x^-\lambda_k^- \tag{2.5}$$

for some l, $1<l<k$.

3. THE SYSTEM OF EQUATIONS FOR THE PERTURBATIONS

We consider a small perturbation of the original displacement front. It corresponds to small deviations of the pressure, densities and fluxes $\mathbf{j}^1$, ρ_i^1, p^1. with respect to these quantities the system of equations has the form

$$\frac{\partial \rho^1}{\partial t}+(j_x^0\mathbf{A}^0-V\mathbf{I})\frac{\partial \rho^1}{\partial x}=0 \tag{3.1}$$

$$\frac{\partial}{\partial x}Q^1+\frac{\partial}{\partial y}j_y^1+\frac{\partial}{\partial z}j_z^1=0,\quad Q^1=j_x^1+b_j^0\rho_j^1 \tag{3.2}$$

$$Q^1=-K^0\frac{\partial p^1}{\partial x}-D^0\rho^1,\quad j_y^1=-K^0\frac{\partial p^1}{\partial y},\quad j_z^1=-K^0\frac{\partial p^1}{\partial z} \tag{3.3}$$

The deviation of the discontinuity from the unperturbed (null) state is denoted by $X^1(t, y, z)$. Using the coefficients introduced in (1.9), the conditions at the discontinuity can be written in the form

$$[\rho^0]\partial X^1/\partial t=[(j_x^0\mathbf{A}^0-V\mathbf{I})\rho^1]+[f^0Q^1] \tag{3.4}$$

$$[\rho_{k+1}^0]\partial X^1/\partial t=[\Phi^{0T}(\mathbf{A}-V\mathbf{I})\rho^1]+[(\Phi^0\cdot f^0)Q^1]+[Q^1] \tag{3.5}$$

$$[p^1]=-[\kappa^0]X^1 \tag{3.6}$$

In Eqs (3.1)–(3.6) the superscripts 0, 1, respectively, refer to the zeroth and first approximations, $\mathbf{I}=[\delta_{ij}]$ is the unit matrix, ρ^0, ρ^1, f^0, Φ^0, D^0 are vectors composed of the coefficients ρ_j^0, ρ_j^1, f_j^0, φ_j^0, D_j^0 $(j=1, 2\dots, k)$, with the last coefficients defined by

$$D_j^0=\kappa_\partial^0 K^0/\partial\rho_j-b_j^0 \tag{3.7}$$

Equations (3.1) and (3.4) are vectorial (systems of k equations), and the first term on the right-hand side of (3.5) is the convolution of two vectors with the matrix $\mathbf{A}-V\mathbf{I}$. Condition (3.6) is a consequence of the pressure continuity equation (1.5) and the "drift" of the discontinuity conditions onto the $x=0$ plane [1].

In system (3.1)–(3.3) the coefficients depend on the quantities $\rho_j^\pm$, $j_x^\pm$ which are discontinuous at $x=0$. Hence (3.1)–(3.3) can be considered to be two systems of linear equations with constant coefficients, coupled to one another through the boundary conditions (3.4)–(3.6) at $x=0$. In addition, it is required that the perturbations tend to zero as $x\to-\infty$ in the domain in front of the discontinuity and as $x\to+\infty$ in the domain behind the discontinuity. The initial conditions for system (3.1)–(3.3) remain undefined.

The displacement front is considered to be unstable a solution of the problem exists which increases without limit as $t\to\infty$.

4. DERIVATION OF THE DISPERSION RELATIONS

The linear hyperbolic system of equations (3.1) is independent of Eqs (3.2) and (3.3). Its general solution is the sum of a steady solution corresponding to a "contact" perturbation which moves with the velocity of the discontinuity, and an unsteady solution for perturbations whose velocity differs from the discontinuity velocity. The first type of perturbation is considered in the following section. The part of the general solution corresponding to an unsteady perturbation has the form

$$\rho^1 = \sum_{\alpha=1,\ \lambda_\alpha^0 \neq V}^{k} d^\alpha \Psi^\alpha\left(t - \frac{x}{j_x^0\lambda_\alpha^0 - V},\ y,\ z\right) \tag{4.1}$$

Here d_α are the right-hand eigenvalues of the matrix $\mathbf{A}^0$, λ_α^0 are its eigenvalues, and Ψ^α are arbitrary functions ($\alpha = 1, 2\ldots, k$). Without loss of generality we choose these functions in the form

$$\Psi^\alpha = C^\alpha(y,\ z)\exp\left\{\omega\left(t - \frac{x}{j_x^0\lambda_\alpha^0 - V}\right)\right\},\quad \omega = \text{const} \tag{4.2}$$

which corresponds to expanding an arbitrary solution ρ^1 as an integral $\int e^{\omega t}\rho^*(\omega,\ x,\ y,\ z)d\omega$ which appears as the result of applying the method of separation of (space and time) variables to system (3.1). The unknown functions in Eqs (3.2)–(3.6) can be represented similarly

$$Q^1 = Q^*(x,\ y,\ z)e^{\omega t},\ p^1 = p^*(x,\ y,\ z)e^{\omega t},\ X^1 = X^*(y,\ z)e^{\omega t} \tag{4.3}$$

We shall assume that system (3.1)–(3.6) has a solution corresponding to an instability of the discontinuity. Then

$$\operatorname{Re}\omega > 0 \tag{4.4}$$

On the other hand, the quantity ρ^1 should tend to zero as $x \to \pm\infty$. By virtue of the Lax conditions (2.5) this leads to removal from the sum in (4.1) of terms corresponding to perturbations incident on the discontinuity.

Using in (4.1) only those terms that vanish as $x \to \pm\infty$ (corresponding to characteristics leaving the discontinuity), we reduce it to the form

$$\begin{aligned} \rho^{*-} &= \sum_{\alpha=1}^{l-1} C^{\alpha-}d^{\alpha-}\exp\left(-\frac{\omega x}{j_x^-\lambda_\alpha^- - V}\right)\quad (j_x^-\lambda_\alpha^- - V < 0) \\ \rho^{*+} &= \sum_{\alpha=l+1}^{k} C^{\alpha+}d^{\alpha+}\exp\left(-\frac{\omega x}{j_x^+\lambda_\alpha^+ - V}\right)\quad (j_x^+\lambda_\alpha^+ - V > 0) \end{aligned} \tag{4.5}$$

Here the number l is the same as in the Lax conditions (2.5).

To solve Eqs (3.2) and (3.3) we Fourier transform them with respect to y, z and substitute the last two relations of (3.3) into (3.2). Keeping the same notation for the transforms as for the original quantities, and using (4.3), we obtain

$$\partial Q^* / \partial x + K^0\gamma^2 p^* = 0,\quad \gamma^2 = \zeta^2 + \eta^2,\quad Q^* = -K^0\partial p^* / \partial x - D^0\rho^*$$

Here ζ and η are the Fourier transform coordinates replacing z and y, respectively.

Solving the resulting system of first-order linear differential equations for the variables Q^*, p^* and rejecting terms that do not tend to zero when $x \to \pm\infty$, we find

$$\begin{aligned} &Q^* = Q^\pm\exp(\mp\gamma x) + \sum_\pm G_\alpha\exp(\omega\beta_\alpha x),\quad p^* = \pm\frac{Q^\pm}{K^0\gamma}\exp(\mp\gamma x) - \sum_\pm \frac{\omega\beta_\alpha}{K^0\gamma^2}G_\alpha\exp(\omega\beta_\alpha x) \\ &G_\alpha = \frac{C_\alpha(D^0 d^\alpha)}{\omega^2\beta_\alpha^2 - \gamma^2},\quad \beta_\alpha = (j_x^0\lambda_\alpha - V)^{-1}\quad (\alpha = 1,\ 2\ldots,\ k) \end{aligned} \tag{4.6}$$

The minus and plus indices in $Q^\pm$ and $\Sigma_\pm$, respectively, denote quantities in front of and behind the discontinuity. The signs in the equations are chosen to correspond to these indices. The summation Σ_+ is performed over α from $l+1$ to k, and the summation Σ_- is performed over α from 1 to $l-1$.

Substituting solutions (4.1)–(4.6) into the discontinuity conditions (3.4)–(3.6) and putting $x=0$ in these equations, after algebraic transformations we obtain the system of equations

$$[\rho_i^0]=\sum_{\alpha=l+1}^{k}\frac{U_\alpha^+ d_\alpha^+}{\beta_\alpha^+}-\sum_{\alpha=1}^{l-1}\frac{U_\alpha^- d_\alpha^-}{\beta_\alpha^-}+f^{0+}j^+-f^{0-}j^- \tag{4.7}$$

$$[\rho_{k+1}^0]=\sum_{\alpha=l+1}^{k}\frac{U_\alpha^+(\Phi^{0+}d^{\alpha+})}{\beta_\alpha^+}-\sum_{\alpha=1}^{l-1}\frac{U_\alpha^-(\Phi^{0-}d^{\alpha-})}{\beta_\alpha^-}+(\Phi^+f^{0+})j^+-(\Phi^-f^{0-})j^-+j^+-j^- \tag{4.8}$$

$$[\kappa^0]=\omega\left(\frac{j^+}{K^+\gamma}+\sum_{\alpha=l+1}^{l-1}U_\alpha^+R_\alpha^+ +\frac{j^-}{K^-\gamma}-\sum_{\alpha=1}^{k}U_\alpha^-R_\alpha^-\right) \tag{4.9}$$

$$U_\alpha^+=\frac{C_\alpha^\pm}{\omega X^*},\quad R_\alpha^\pm=\frac{(D^{0\pm}d^{\alpha\pm})}{K^\pm\gamma^2(\omega\beta_\alpha^\pm\pm\gamma)},\quad j^\pm=\frac{1}{\omega X^*}\left(Q^\pm+\sum_\pm\frac{U_\alpha^\pm(D^0d^{\alpha\pm})}{\omega^2\beta_\alpha^{\pm2}-\gamma^2}\right) \tag{4.10}$$

When deriving Eqs (4.7) and (4.8) we used the fact that the vectors $d^{\alpha\pm}$ are eigenvectors of the matrices $\mathbf{A}^\pm$. The expressions $(\Phi^{0+}d^{\alpha+})$, $(\Phi^{0-}d^{\alpha-})$, (Φ^+f^{0+}), (Φ^-f^{0-}), $(D^{0+}d^{\alpha+})$, $(D^{0-}d^{\alpha-})$ are the scalar products of the vectors they contain.

System (4.7), (4.8) should be considered as $k+1$ linear equations in the $k+1$ unknowns j^+, j^-, $U_\alpha^\pm$ ($\alpha=1,2,\ldots,k$; $\alpha\neq l$). Solving these equations and substituting the solution into (4.9), we obtain an algebraic equation in ω. the front is unstable if for some γ this equation has a solution such that $\mathrm{Re}\,\omega>0$.

The solution simplifies if the total flux is incompressible and the original equation (1.2) has the form $\operatorname{div}\mathbf{j}=0$.

In that case Eqs (4.7) and (4.9) remain unchanged, while (4.8) is replaced by the condition

$$j^+=j^- \tag{4.11}$$

5. STABILITY WITH RESPECT TO CONTACT PERTURBATIONS

Let $d^{\upsilon+}$, $d^{\upsilon-}$ be vectors satisfying the equations

$$(\mathbf{A}^+-V\mathbf{I})d^{\upsilon+}=(\mathbf{A}^--V\mathbf{I})d^{\upsilon-}=0 \tag{5.1}$$

If the velocity of the discontinuity V is an eigenvalue of the matrix $\mathbf{A}^+$ ($\mathbf{A}^-$), then $d^{\upsilon+}$ ($d^{\upsilon-}$) is the corresponding eigenvector. In this case a contact characteristic exists coinciding with the discontinuity. In the opposite case the vector $d^{\upsilon+}$ ($d^{\upsilon-}$) vanishes.

As well as solutions of the form (4.1), the system of equations (3.1)–(3.3) has a solution that is steady in a frame of reference attached to the discontinuity

$$\rho^{1\upsilon}=d^{\upsilon\pm}\Psi_1^\pm(x,\ y,\ z)+\Psi_2^\pm(y,\ z) \tag{5.2}$$

As above, without loss of generality we put

$$\Psi_1^\pm=C^\pm(y,\ z)\exp(\beta^\pm x) \tag{5.3}$$

In order to satisfy the conditions as $x\to\infty$ one must put $\beta^->0$, $\beta^+<0$, $\Psi_2^\pm=0$.

In this case the quantities p^1 and Q^1 have a form similar to (4.6)

$$Q^{1\upsilon} = Q^{\pm}\exp(\mp\gamma x) + H^{\pm}\exp(\beta^{\pm}x),\ p^{1\upsilon} = \mp\frac{1}{K^{0\pm}\gamma}Q^{\pm}\exp(\mp\gamma x) - H^{\pm}\exp(\beta^{\pm}x)$$

$$Q^{\pm} = Q^{\pm}(t,\ y,\ z),\ H^{\pm}\frac{C^{\pm}(D^{0\pm}d^{\upsilon\pm})}{\beta^{\pm 2} - \gamma^{2}}$$

Substituting these equations into the discontinuity conditions (3.4)–(3.6), using a replacement similar to (4.10), and $(\mathbf{A} - V\mathbf{I})\rho^{*} = 0$ which holds because of (5.1) and (5.2), we obtain the system

$$[\rho_i^0]\frac{\partial X^1}{\partial t} = f^{0+}j^{+} - f^{0-}j^{-} \tag{5.4}$$

$$[\rho_{k+1}^0]\frac{\partial X^1}{\partial t} = j^{+} - j^{-} + (\varphi^{0+}f^{0+})j^{+} - (f^{0-}\varphi^{0-})j^{-} \tag{5.5}$$

$$[\kappa^0]X^1 = \frac{j^{+}}{K^{0+}\gamma} + \frac{j^{-}}{K^{0-}\gamma} + \chi^{+} - \chi^{-},\ \chi^{\pm} = \frac{(D^{0\pm}d^{\upsilon\pm})C^{\pm}}{K^{0\pm}\gamma^{2}(\beta^{\pm} \pm \gamma)} \tag{5.6}$$

Note that the system of equations (5.4), (5.5) is identical with the system of conditions at the discontinuity in the zeroth approximation (2.4), with V replaced by $\partial X^1/\partial t$ and $j_x^{0\pm}$ replaced by $j^{\pm}$. Hence, assuming that the rank of the matrix

$$\begin{bmatrix} f_0^{+} & f_0^{-} \\ 1+(\Phi^{0+}f^{0+}) & 1+(\Phi^{0-}f^{0-}) \end{bmatrix}$$

is equal to two, this system has the unique solution

$$j^{\pm} = j_x^{0\pm}V^{-1}\partial X^1/\partial t$$

Substituting this solution into (5.6), we obtain the differential equation

$$\frac{\partial X^1}{\partial t} + \lambda X^1 = C,\ \lambda = \frac{-V[\kappa^0]\gamma}{j_x^{0+}/K^{0+} + j_x^{0-}/K^{0-}} = \frac{V\gamma(\kappa^{-} - \kappa^{+})}{\kappa^{-} + \kappa^{+}},\ C = \frac{V\gamma(\chi^{-} - \chi^{+})}{\kappa^{-} + \kappa^{+}} \tag{5.7}$$

The quantity λ is constant while C depends of y and z. Equation (5.7) has the solution

$$X^1 - C = (X^1(0) - C)\exp(-\lambda t) \tag{5.8}$$

which is bounded when $t \to \infty$ if $\lambda > 0$.

We note that Eq. (5.7) remains true (with the constant C zero) if there are no contact characteristics. In that case the condition $\lambda > 0$ is the criterion for the stability of the discontinuity with respect to pressure and flux perturbations without density perturbations. When there are no contact characteristics C is zero, and the discontinuity perturbations decay at infinity. In the opposite case they tend to a non-zero constant value.

In the often encountered special case when j_x^{0-}, j_x^{0+} and V have the same sign, the stability condition relative to contact perturbations has the usual form $[\kappa^0] < 0$ [5, 7]: the motion is stable if the pressure gradient in the displaced fluid is less than in the displacing fluid. Using expressions (2.3) for κ^0 we can write this condition in the form

$$j_x^{0+}/K^{0+} < j_x^{0-}/K^{0-} \tag{5.9}$$

If the flow is incompressible and positive, this last equation acquires the form

$$K^{0-} < K^{0+} \tag{5.10}$$

i.e. the displacement front is stable if the mobility of the displacing phase is lower than that of the displaced phase.

As an example we consider flows described by two equations of the form (1.1), (1.2) ($k = 1$). These include, for example, two-phase flow of immiscible liquids (and in the special case of incompressible phases this is Buckley–Leverett flow) and two-phase two-component flow in a porous medium [5, 9]. Because of the Lax conditions there are no characteristics leaving discontinuities of such flows. Hence the stability of the discontinuities is governed by the condition $\lambda > 0$. For one-dimensional (plane-parallel or plane-radial) flows the assumptions that lead to condition (5.10) hold. Here the quantities K^{0+}, K^{0-} are to be interpreted as the total mobilities of the phases before and after the discontinuity.

Thus, to verify the stability of a discontinuity with respect to contact perturbations it is necessary to check the condition $\lambda > 0$. The condition for stability with respect to perturbations that leave the discontinuity is the positivity of the quantity ω defined by system (4.7)–(4.9). Perturbations incident on the discontinuity do not influence its stability.

6. STABILITY OF DISPLACEMENT BY A SOLUTION WITH AN ACTIVE ADMIXTURE

We apply the formalism developed above to the stability of a discontinuity associated with the concentration jump in the self-similar problem of the displacement of oil by a solution with an active admixture. The system of equations describing this process has the form [5, 10]

$$m\frac{\partial S}{\partial t} + \operatorname{div}(F\mathbf{j}) = 0, \quad m\frac{\partial CS}{\partial t} + \operatorname{div}(CF\mathbf{j}) = 0 \tag{6.1}$$

$$\operatorname{div}\mathbf{j} = 0, \quad \mathbf{j} = -\Pi\nabla p, \quad \Pi = \frac{f_0(S,\ C)}{\mu_0(C)} + \frac{f_w(S,\ C)}{\mu_w(C)} \tag{6.2}$$

Here S is the saturation of the water phase, C is the mass concentration of the admixture, $F = F(S, C)$ is the Buckley–Leverett function, Π is the total mobility of the phase, f_0, f_w are the phase permeability functions and μ_0, μ_w are the dynamic viscosities of oil and water, respectively. It is assumed that the admixture is transported only by the water flux and is not sorbed by the rock skeleton. The more general case of a sorbable admixture which is soluble in oil can be treated similarly.

We consider one-dimensional solutions of system (6.1), (6.2). In this case the system can be simplified. It follows from the first equation of (6.2) that the flux $\mathbf{j}$ depends only on time. Introducing the new variable $\tau = m^{-1}\int j\,dt$ we reduce Eq. (6.1) to the form

$$\frac{\partial S}{\partial \tau} + \frac{\partial F}{\partial x} = 0, \quad \frac{\partial C}{\partial \tau} + \frac{\partial CF}{\partial x} = 0 \tag{6.3}$$

Boundary conditions corresponding to the displacement of oil by the active admixture solution have the form

$$\tau = 0 : S = S^{+}, C = 0; \; x = 0 : F = 1, S = S^{-}, C = C^{-}$$

System (6.3) with the given boundary condition have a self-similar solution depending on the single variable $\xi = x/\tau$. This solution is constructed grapho-analytically using the (S, F) plane (see Fig. 1) [5, 8].

The solution constructed, corresponding to the case of high initial water-saturation S^{+}, contains two discontinuities. One consists of a simple S-wave (section 1–2), a concentration

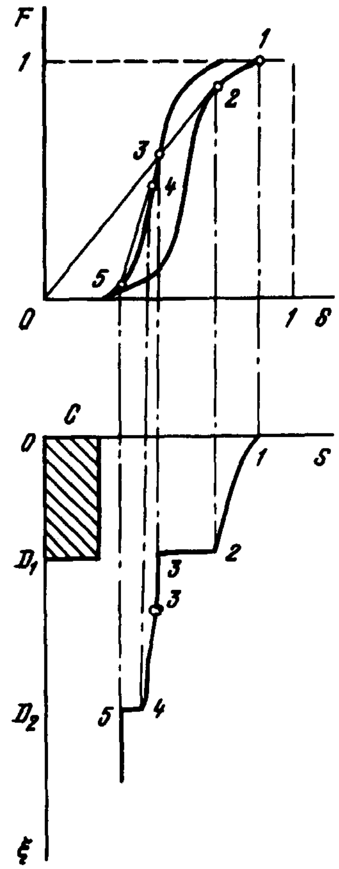

Fig. 1.

jump D_1 (section 2–3), a degenerate wave (section 3–3), then another simple S-wave (section 3–4), a water-saturation jump D_2, and another degenerate wave (section 4–5). As a result of substituting $\xi = x/\tau$ into the first equation of (6.3) we arrive at the expression $\xi = dF(S,\ C)/dS$ which means that in the (S, F) plane the self-similar variable ξ is identical with the angular coefficient of the tangent to the $F(S, C(S))$ curve [5].

The water-saturation jump D_2 is constructed as in the Buckley–Leverett problem [5]. Because this problem is described by two equations of the form (1.1), (1.2), the only stability condition for this jump is the mobility ratio (5.10). Unlike the case of the D_2 jump, to clarify the stability question for the D_1 concentration jump it is necessary to investigate its stability with respect to non-contact perturbations.

System (6.1), (6.2) is a special case of the system of equations (1.9) in which one must put

$$\mathbf{A} = \begin{bmatrix} F_s' & F_c' \\ 0 & F/S \end{bmatrix}, \quad B = 0, \quad \varphi_p = b_j = 0$$

The relation between systems (6.1), (6.2) and (1.9) allows one to find linear equations in the first approximation for the perturbed motion of the discontinuity front (3.1)–(3.3), where

$$\rho^1 = \begin{pmatrix} S \\ C \end{pmatrix}, \quad Q^1 = -\Pi^0 \frac{\partial p}{\partial X} - D_s^0 S - D_c^0 C, \quad D_s^0 = \frac{j_x^0}{\Pi^0}\frac{\partial \Pi^0}{\partial S}, \quad D_c^0 = \frac{j_x^0}{\Pi^0}\frac{\partial \Pi^0}{\partial C}$$

The system of linear algebraic equations at the D_1 jump for the Fourier-transforms with respect to y and z has, according to (4.7)–(4.10), the form

$$S^+ - S^- = -U(F^+ / S^+) j_x^0 (F_s'^+ - F_s'^-) + (F^+ - F^-) j \tag{6.4}$$

$$-C^- = U F_c'^+ j_x^0 (F_s'^+ - F_s'^-) - C^- F^- j \tag{6.5}$$

$$\frac{j_x^0}{\Pi^+} - \frac{j_x^0}{\Pi^-} = \omega\left(\frac{j}{\Pi^+\gamma} - \frac{j}{\Pi^-\gamma} + U\left(D_c^0 F_c'^+ - D_s^0 \frac{F^+}{S^+}\right)\left(\Pi^+\gamma^2\left(\frac{\omega}{j_x^0(F_s'^+ - F_s'^-)} + \gamma\right)\right)^{-1}\right) \tag{6.6}$$

Solving (6.4) and (6.5) for the constants U and j and then substituting into (6.6) we obtain a quadratic equation for the perturbation decrement ω. Omitting complicated transformations and taking the jump conditions into account [5]

$$F_s'^+ > F_s'^-, \quad F^+ < F^-, \quad S^+ < S^-, \quad F_c' < 0, \quad F_s' > 0$$

$$(F^+ - F^-)/(S^+ - S^-) > 1$$

we find from the Routh–Hurwitz criteria [11] that the necessary and sufficient conditions of stability for all wave numbers γ are the inequalities

$$\Pi^- < \Pi^+ \tag{6.7}$$

$$\frac{(S^+ - S^-)F_c'^+ - (F^+ C^- / S^+)}{(S^+ - S^-)F_c'^+ - C^-} > 0 \tag{6.8}$$

$$\frac{(S^+ - S^-)F_c'^+ - (F^+ C^- / S^+)}{(S^+ - S^-)F_c'^+ - C^-} < \frac{F^+}{(F_s'^+ - F_s'^-)S^+} \tag{6.9}$$

$$\frac{F_c'^+(\partial\Pi^+ / \partial C) - (F^+ / S^+)(\partial\Pi^+ / \partial S)}{(S^+ - S^-)F'^+ - C^- F^-} > 0 \tag{6.10}$$

Only one of these inequalities need be violated for the jump to be unstable. Thus, for the concentration jump to be stable, as well as the usual mobility relation (6.7) it is also necessary that the additional conditions (6.8)–(6.10) should be satisfied. These conditions are non-trivial and independent of (6.7). As an example we consider condition (6.9). We assume that the concentration $C = C^-$ is small. Then on the left-hand side of this condition there is a quantity close to unity. Because the ratio of the derivatives of the Buckley–Leverett functions before and after the jump is almost arbitrary (we only know that $F_s'^+ > F_x'^-$), the right-hand side of inequality (6.9) may turn out to be a number which can be larger or much smaller than unity.

In just the same way conditions (6.8) and (6.10) can be satisfied or violated, even if the mobility ratio (6.7) indicates that the jump is stable.

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